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# Shear flow instabilities in rotating systems

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A theoretical description is given for infinitesimal non-axisymmetric disturbances of a shear flow caused by differential rotation. It is assumed that the deviations from the state of rigid rotation are small corresponding to the case of small Rossby number. The shear flow becomes unstable at a finite critical value of the Rossby number, at which the inertial forces overcome the friction in the Ekman boundary layers and the constraint imposed by the variation in depth of the container. The governing equation is closely related to the Rayleigh stability equation in the theory of hydrodynamic stability of plane parallel flow. Experimental observations by R. Hide show reasonable agreement with the theoretical predictions.

## 1. Introduction

The dynamics of a homogeneous fluid in a rotating system are in many respects simpler than in a corresponding non-rotating system. Boundary-layer theory has a wider range of application in a rotating system, and viscous dissipation can be taken into account more easily. This fact holds in particular for the theory of hydrodynamic stability, and one of the motivations for this paper is to show that the extended literature on the inviscid theory of stability has a direct application to flows in rotating systems. In non-rotating systems dissipative effects usually necessitate a more complicated description.

We shall restrict the stability analysis to steady flow, which is axisymmetric with respect to the axis of rotation, and assume that the vorticity of the flow is small compared with the rotation rate of the system; i.e. we consider cases with small Rossby number. According to the Taylor–Proudman theorem, axisymmetric steady flow of small Rossby number assumes the form of a differential rotation.

Shear flow occurring in the form of differential rotation is a frequent phenomenon in rotating fluid systems. An example is the cylindrical shear layers formed at the junction between two fluid bodies rotating rigidly about the same axis but at different rates. Free stationary shear layers of this nature have been described mathematically by Stewartson (1957), who studied the motion of a fluid between two infinite parallel rotating plates when a circular part of one or both plates is rotating at a different rate. In general, a differential rotation with arbitrary dependence on the distances from the axis can be produced in an axisymmetric system when the bounding surfaces are rotating differentially.

Often stationary shear flow is induced by forced oscillations of the fluid. The non-linear terms in the equation of motion lead to a rectified component which has the form of a differential rotation in an axisymmetric system. An example for this type of shear flow is the flow in a precessing spherical shell which has been studied by Malkus (1968) and Busse (1968).

In general, the laminar azimuthal shear flow becomes unstable when its amplitude increases sufficiently. Two basic mechanisms for instability can be distinguished. When the angular velocity $\hat{\omega}$ with respect to an inertial system decreases with distance $s$ from the axis, the Rayleigh criterion $d\hat{\omega}s^2/ds > 0$ with respect to axisymmetric disturbances (Rayleigh 1920) can be violated. The Taylor vortices in the case of circular Couette flow are an example for this kind of instability. A different kind of instability occurs as a wave propagating in the azimuthal direction and corresponds to the instability of parallel shear flow in a non-rotating system. We shall restrict the discussion to the latter type of instability since the Rayleigh criterion is satisfied for sufficiently small Rossby numbers.

To formulate the mathematical problem we consider a rotating system with an incompressible homogeneous fluid contained between rigid boundaries. Given the constant rotation rate $\Omega$, the kinematic viscosity $\nu$, and a characteristic length $L$ of this system in the direction of the axis of rotation, a dimensionless number $E = \nu/L^2\Omega$, the Ekman number, can be formed. We use $L$ as the length scale and $\Omega^{-1}$ as the scale for time in order to obtain dimensionless variables. We further assume that the Navier–Stokes equations of motion describing the system have a steady solution of the form

$$\mathbf{U} = \mathbf{k} \times \mathbf{r}\epsilon f(s). \tag{1.1}$$

$\mathbf{U}$ is the velocity vector, $\mathbf{k}$ is the unit vector parallel to the axis of rotation, and $\mathbf{r}$ is the position vector. $s$ is equal to $|\mathbf{k} \times \mathbf{r}|$. The amplitude of the flow (1.1) is given by the Rossby number $\epsilon$, since we assume that the otherwise unspecified function $f$ is of the order one. A steady solution of the form (1.1) can be expected in the limit of vanishing $E$ when the additional advective flow balancing the viscous diffusion become negligible. In many situations the solution (1.1) is also modified in Ekman boundary layers close to the rigid boundaries. We shall neglect this modification since cases are feasible in which the boundary velocities are chosen in such a way that the modification of (1.1) in the Ekman layer becomes arbitrarily small. We expect, however, as we shall discuss in §6, that the influence of the modification can be neglected even when its amplitude is of the order $\epsilon$.

The mathematical problem is to determine, as a function of the other parameters of the system, the ciritical value of $\epsilon$ at which the solution (1.1) becomes unstable with respect to disturbances of infinitesimal amplitudes. We expect a strong influence of the change in depth of the fluid in the direction of the axis of rotation, since it is known from the general analysis by Greenspan (1965) of free oscillations in rotating containers that systems with constant depth in the $\mathbf{k}$-direction are exceptional. The instability in systems of this kind is discussed in §2. The analysis is extended to more general cases in §3. In the limit when the steady flow can be approximated by plane parallel shear flow, the analysis leads to well-known equations. For this reason an explicit solution of the problem is given in §5 for two special cases only, which exhibit the effects of the circular geometry. The case of strongly varying depth requires a more involved analysis. In §4 the stability problem is formulated, and a special solution is derived to demonstrate some characteristic features of this case. The relevance of the theory to the experimental situation is considered in the final §§6 and 7.

## 2. Rotating systems with constant depth

The Navier–Stokes equations for an infinitesimal disturbance $\mathbf{q}$ of the given velocity field (1.1) have the following dimensionless form:

$$\left.\begin{aligned} E\nabla^2\mathbf{q}-\nabla p-2\mathbf{k}\times\mathbf{q} &= \mathbf{U}.\nabla\mathbf{q}+\mathbf{q}.\nabla\mathbf{U}+\frac{\partial}{\partial t}\mathbf{q},\\ \nabla.\mathbf{q} &= 0. \end{aligned}\right\} \tag{2.1}$$

Using polar co-ordinates $(s,\phi,z)$ with respect to the $\mathbf{k}$-axis, we assume that the normal unit vector of the bounding surfaces is given by $\mathbf{n} = (n_s(s), 0, n_z(s))$. In this section we further assume that the fluid has a constant depth, implying that the outward-pointing normal vectors $\mathbf{n}^T$ at the top and $\mathbf{n}^B$ at the bottom surface have identical dependence on $s$ but opposite signs. The boundary condition on these boundaries is

$$\mathbf{q} = 0. \tag{2.2}$$

In addition to the top and bottom surfaces, for which $n_z$ is non-vanishing, side-walls parallel to $\mathbf{k}$ can be admitted. On the side-walls only the normal component of $\mathbf{q}$ is required to vanish. The condition that the remaining components of $\mathbf{q}$ have to vanish can be shown in most cases to have negligible influence.

Using the assumption that $\sqrt{E}$ and $\epsilon$ are small parameters, we solve equation (2.1) by applying the boundary-layer method as described, for example, in Greenspan's (1965) paper. The velocity field is separated into two parts which both are expanded in powers of $\sqrt{E}$,

$$\mathbf{q} = \mathbf{q}_i+\tilde{\mathbf{q}} = \mathbf{q}_0+\sqrt{E}\,\mathbf{q}_1+\ldots+\tilde{\mathbf{q}}_0+\sqrt{E}\,\tilde{\mathbf{q}}_1. \tag{2.3}$$

$\mathbf{q}_i$ describes the velocity field throughout the interior, while $\tilde{\mathbf{q}}$ is non-vanishing only in a thin layer close to the boundary.

The linear equations (2.1) allow a time dependence of the form $\exp\{-i\epsilon\omega t\}$. The phase velocity will be of the order of the Rossby number because of Howard's semicircle theorem which has been extended to circular flow by Eckart (1963). Since we are interested in the point of marginal stability, we have to determine the functional relation between $\epsilon$ and $\sqrt{E}$ for which $\omega$ becomes real. We anticipate that $\epsilon$ will be of the order $\sqrt{E}$ in this case, corresponding to a balance between the viscous dissipation in the boundary layer and the forcing term due to the shear flow. Neglecting terms of the order $\epsilon$ or $\sqrt{E}$ in (2.1), we arrive at the following equations for $\mathbf{q}_0$:

$$2\mathbf{k}\times\mathbf{q}_0+\nabla p_0 = 0, \quad \nabla.\mathbf{q}_0 = 0, \tag{2.4}$$

which imply the Taylor–Proudman theorem

$$\mathbf{k}.\nabla\mathbf{q}_0 = 0. \tag{2.5}$$

The general solution of (2.4) which satisfies the inviscid part

$$\mathbf{n}.\mathbf{q}_0 = 0 \tag{2.6}$$

of the boundary condition (2.2) is given by

$$\mathbf{q}_0 = \frac{1}{2n_z^T}\mathbf{n}^T \times \nabla p_0. \tag{2.7}$$

In order to satisfy the equation of continuity, we assume that the dependence of $\mathbf{n}^T$ on $s$ is small compared with that of $p_0$. $p_0$ is an undetermined function of $s, \phi$ which vanishes at the side-walls $s = s_i, s_0$ where $s_i, s_0$ may have the values zero and infinity. In order to satisfy the complete boundary condition (2.2) the boundary-layer problem has to be solved. By introducing the co-ordinate $\zeta$ in the direction normal to the boundary,

$$\zeta = -E^{-\frac{1}{2}}\mathbf{n}.(\mathbf{r}-\mathbf{r}_s),$$

with $\mathbf{r}_s$ denoting the position vector at the boundary, the zeroth-order boundary-layer equation can be written

$$\frac{\partial^2}{\partial\zeta^2}\tilde{\mathbf{q}}_0 + \mathbf{n}\frac{\partial}{\partial\zeta}\tilde{p}_1 = 2\mathbf{k}\times\tilde{\mathbf{q}}_0. \tag{2.8}$$

The general solution of this equation with the boundary condition $\tilde{\mathbf{q}}_0 + \mathbf{q}_0 = 0$ is of interest in our case only in so far as it leads to an influx into the boundary due to the equation of continuity (Greenspan 1965)

$$\mathbf{n}.\tilde{\mathbf{q}}_1|_{\zeta=0} = \tfrac{1}{2}\mathbf{n}.\nabla\times\left\{\left(\mathbf{n}\times\mathbf{q}_0 + \frac{n_z}{|n_z|}\mathbf{q}_0\right)|n_z|^{-\frac{1}{2}}\right\}. \tag{2.9}$$

To balance this influx the equation for $\mathbf{q}_1$,

$$2\mathbf{k}\times\mathbf{q}_1 + \nabla p_1 = -E^{-\frac{1}{2}}[\mathbf{U}.\nabla\mathbf{q}_0 + \mathbf{q}_0.\nabla\mathbf{U} - i\epsilon\omega\mathbf{q}_0] \tag{2.10}$$

has to be solved subject to the boundary condition

$$\mathbf{n}.\mathbf{q}_1 + \mathbf{n}.\tilde{\mathbf{q}}_1|_{\zeta=0} = 0. \tag{2.11}$$

The solvability condition for the inhomogeneous problem (2.10), (2.11) is crucial in the determination of the unknown pressure field $p_0$. Let

$$\mathbf{q}_0^* \equiv \frac{1}{2n_z^T}\mathbf{n}^T\times\nabla p_0^* \tag{2.12}$$

be a solution of (2.4) satisfying the same boundary conditions as $\mathbf{q}_0$. By multiplying (2.10) with $\mathbf{q}_0^*$, integrating over the interior region, and using $\nabla.\mathbf{q}_1 = 0$, we obtain

$$\oint\{\mathbf{n}.\mathbf{q}_1\}p_0^*\,d\Sigma = -E^{-\frac{1}{2}}\int\mathbf{q}_0^*.[\mathbf{U}.\nabla\mathbf{q}_0 + \mathbf{q}_0.\nabla\mathbf{U} - i\epsilon\omega\mathbf{q}_0]\,dV$$
$$= +\int\frac{2}{n_z^T}p_0^*\left\{\frac{1}{4\sqrt{E}}\mathbf{n}^T.\nabla\times[\mathbf{U}.\nabla\mathbf{q}_0 + \mathbf{q}_0.\nabla\mathbf{U} - i\epsilon\omega\mathbf{q}_0]\right\}dV \tag{2.13}$$

as a necessary and sufficient condition for the solvability of (2.10). For $p_0^* = p_0$ the relation (2.13) can be interpreted as the energy balance for $\mathbf{q}_0$. Since the integration over the $z$ co-ordinate contributes the factor 1 and because the relation (2.13) has to hold for arbitrary function $p_0^*$ of $s, \phi$, the content of the wavy brackets on both sides of (2.13) has to be equal. This fact provides a differential equation for $p_0$ which we write in the form

$$(f-c)\left((n_z^T)^2\frac{1}{s}\frac{d}{ds}s\frac{d}{ds}-\frac{m^2}{s^2}\right)\chi-(n_z^T)^2\left(\frac{3}{s}f'+f''\right)\chi=0, \tag{2.14}$$

since

$$p_0=\chi(s)\exp\{i(m\phi-\epsilon\omega t)\} \tag{2.15}$$

can be assumed without losing generality. The constant $c$ is defined by

$$c=\frac{\omega+2i\epsilon^{-1}E^{\frac{1}{2}}(n_z^T)^{-\frac{1}{2}}}{m}. \tag{2.16}$$

Equation (2.14) resembles the equation for the instability of inviscid shear flow in a non-rotating system, since the influence of rotation appears only in the form of the dissipative term proportional to $\sqrt{E}$. In fact, in the limit when

$$(s_0-s_i)/(s_0+s_i)\ll 1, \tag{2.17}$$

or when $f$ differs from a constant only over a corresponding small interval, the effects of the circular geometry become negligible and (2.14) can be rewritten in the form

$$(f-c)\left(\frac{d^2}{ds^2}-\alpha^2\right)\chi-f''\chi=0, \tag{2.18}$$

with

$$\alpha=\frac{2m}{(s_i+s_0)\,n_z^T},$$

which is identical with the well-known Rayleigh stability equation for plane parallel flow. A recent review of the extended literature concerned with (2.18) has been given by Drazin & Howard (1966). Solutions are known for various types of shear flow profiles $f$ and can be applied to the present problem. The maximum of the imaginary part of $mc$ as a function of $m$ determines the critical value $\epsilon_c$ of the amplitude $\epsilon$ at the point of marginal stability for a given profile $f$ and a given value of $E$. In order to demonstrate the influence of the circular geometry, solutions of (2.14) will be derived in §5 in two simple cases.

## 3. Extension of the analysis to more general cases

In the case when the depth of the fluid is varying slightly in the sense that

$$|\mathbf{n}^T+\mathbf{n}^B|\ll 1 \tag{3.1}$$

at corresponding points of top and bottom boundaries (at least in so far as the region where the instability occurs), the analysis of the preceding section can be applied with a small modification. Assuming that $\mathbf{q}_0$ is given in the form

$$\mathbf{q}_0=\frac{1}{2(n_z^T-n_z^B)}(\mathbf{n}^T-\mathbf{n}^B)\times\nabla p_0,$$

we obtain

$$\mathbf{n}.\mathbf{q}_1+\mathbf{n}.\tilde{\mathbf{q}}_1|_{\zeta=0}=\frac{1}{2\sqrt{E}(n_z^T-n_z^B)}(\mathbf{n}^T\times\mathbf{n}^B).\nabla p_0 \tag{3.2}$$

as a boundary condition for $\mathbf{q}_1$ in place of (2.10). The additional term in (3.2) leads to a modified equation for $\chi$:

$$(f-c)\left(\frac{1}{s}\frac{d}{ds}s\frac{d}{ds}-\frac{4m^2}{s^2(n_z^T-n_z^B)^2}\right)\chi+\frac{2}{s\epsilon}\left(\frac{n_s^T}{n_z^T}-\frac{n_s^B}{n_z^B}\right)\chi-\left(\frac{3}{s}f'+f''\right)\chi=0. \quad (3.3)$$

In the limit (2.17) this equation becomes formally identical with the Rayleigh stability equation for parallel shear flow in the $\beta$-plane approximation. The $\beta$-plane approximation has been introduced in the studies of the dynamics of the atmosphere and the oceans to take into account the variation of the Coriolis force. It is well known that slight variations in depth cause a stretching of vortex lines similar to the effect of varying Coriolis force. The analysis of (3.3) in the case of plane parallel flow by Kuo (1949) and Howard & Drazin (1964) shows that the additional term in (3.3) in general has a stabilizing effect.

The analysis of the preceding section can also be extended to include the effect due to viscous dissipation in the interior. Since it has been shown that the boundary analysis holds for cases when $\chi'(s)$ becomes of the order $E^p\chi$ with $0>p>-\frac{1}{2}$ (see, for example, Stewartson, 1957), the dissipative term in the interior can be taken into account in the equation for $\mathbf{q}_1$. The solvability condition leads to

$$\left(f-c-\frac{E}{\epsilon im}\nabla^2\right)\left(\frac{1}{s}\frac{d}{ds}s\frac{d}{ds}-\frac{m^2}{(n_z^T)^2s^2}\right)\chi-\left(\frac{3}{s}f'+f''\right)\chi=0 \quad (3.4)$$

in place of (2.14) in this case. (3.4) corresponds to the Orr–Sommerfeld equation in the limit (2.17).

## 4. Instabilities in the case of strongly varying depth

For simplicity we shall assume in this section that top and bottom surface are symmetric with respect to the plane $z=0$, i.e. $n_z^B=-n_z^T$, $n_s^B=n_s^T$. This special case exhibits the characteristic influence of a strongly varying depth, and the analysis of the general case follows in direct analogy. In the preceding section we have shown that the variation in depth can be taken into account as a perturbation when $\eta\equiv n_s^T/n_z^T$ is of the order $\epsilon$ or $\sqrt{E}$, respectively. In the case of plane parallel flow corresponding to the limit (2.17) of (3.3), Kuo (1949) has shown that a necessary condition for instability is that there exists a point $s_1$ with

$$\epsilon s_1 f''(s_1)=\eta. \quad (4.1)$$

This condition indicates that the onset of instability occurs at a finite Rossby number $\epsilon$ even in the limit of vanishing Ekman number. Although the criterion (4.1) has been derived only for small $\eta$, it suggests that in the case when $\eta$ becomes of the order 1 either the critical Rossby number will become of the order 1, or the characteristic scale of the profile $f$ has to be sufficiently small (i.e. of the order $\sqrt{\epsilon}$) to allow instability. Since we assume that $\epsilon$ is a small parameter we shall restrict our attention to the latter case.

Neglecting the effects due to the circular geometry, we introduce as new coordinates

$$x\equiv\frac{-\phi s_1}{\sqrt{\epsilon}},\quad y\equiv\frac{s-s_1}{\sqrt{\epsilon}}.$$

We note that the $x, y$ co-ordinates are based on $\sqrt{\epsilon}\,L$ as length scale while the $z$ co-ordinate is still based on $L$. In order to derive convenient equations from the basic equations (2.1) we represent the velocity field by

$$\mathbf{q} = \nabla \times (\nabla \times \mathbf{k}\theta) + \nabla \times \mathbf{k}\psi.$$

A representation of this form holds for arbitrary non-divergent velocity fields. Neglecting the viscous effects we derive the following two equations for $\theta$ and $\psi$ from (2.1):

$$\left.\begin{aligned} i\gamma[w(y)-c]\left(\frac{\partial^2}{\partial y^2}-\gamma^2\right)\sqrt{\epsilon}\,\theta+2\frac{\partial}{\partial z}\psi &= 0,\\ i\gamma[w(y)-c]\left(\frac{\partial^2}{\partial y^2}-\gamma^2\right)\psi-i\gamma w''(y)\psi-2\sqrt{\epsilon}\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial y^2}-\gamma^2\right)\theta &= 0.\end{aligned}\right\} \tag{4.2}$$

We have used the definition $w(y) = -s_1 f(s)$ and assumed that the $x$- and $t$-dependence of $\theta$ and $\psi$ is described by $\exp\{i\gamma(x-\epsilon ct)\}$. The assumption that the characteristic scale of the problem in the directions perpendicular to the $\mathbf{k}$-axis is of the order $\sqrt{\epsilon}$ smaller than in the $\mathbf{k}$-direction has allowed us to neglect terms of the order $\sqrt{\epsilon}$ smaller than those retained in (4.2). From (4.2) a single equation for $\psi$ can be derived:

$$[w(y)-c]\left(\frac{\partial^2}{\partial y^2}-\gamma^2\right)\psi-w''(y)\,\psi-\frac{4}{\gamma^2[w(y)-c]}\frac{\partial^2}{\partial z^2}\psi = 0. \tag{4.3}$$

Similarly, the boundary condition for $\psi$ follows from the condition that the normal component of the velocity vanishes at the boundary,

$$\eta\gamma^2[w(y)-c]\,\psi+2\frac{\partial}{\partial z}\psi = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{4.4}$$

In general the problem defined by (4.3) and (4.4) does not allow separation of variables and thus prohibits simple solutions. In order to exhibit the influence of strongly varying depth in a special case, we choose

$$w(y) = [\delta(y)]^{\frac{1}{2}}. \tag{4.5}$$

Although this profile is rather singular it reproduces correctly the dispersion relation $c(\gamma)$ for arbitrary smooth profiles with

$$w(\infty) = w(-\infty) = 0, \quad \text{and the same momentum flux,} \quad \int_{-\infty}^{\infty} w^2(y)\,dy = 1,$$

as long as $\gamma^{-1}$ is large compared with the characteristic width of the 'jet'. This has been shown by Drazin & Howard (1962, 1966), who also derived the relations

$$\lim_{y\to 0+}\psi = \lim_{y\to 0-}\psi, \quad c^2\left(\lim_{y\to 0+}\frac{\partial}{\partial y}\psi-\lim_{y\to 0-}\frac{\partial}{\partial y}\psi\right) = \gamma^2\psi(0), \tag{4.6}$$

which $\psi$ has to satisfy in this case and which are not altered by the addition of the last term on the left-hand side of (4.3). Since the scale of the profile (4.5) is infinitesimal compared with the scale we have chosen, marginally stable disturbances cannot be expected. Thus the scope of the following discussion can only be to show by comparison with vanishing or small values of $\eta$ the influence exerted by finite values of $\eta$.

The profile (4.5) allows us to solve the problem (4.3) and (4.4) by

$$\psi = \exp\left\{i\gamma(x-\epsilon ct) - |y|\left(\gamma^2 - \frac{4\lambda^2}{\gamma^2c^2}\right)^{\frac{1}{2}}\right\}\cos\lambda z, \tag{4.7}$$

where the complex number $\lambda$ is determined by the relation

$$\lambda tg(\tfrac{1}{2}\lambda) = -\tfrac{1}{2}\eta\gamma^2 c. \tag{4.8}$$

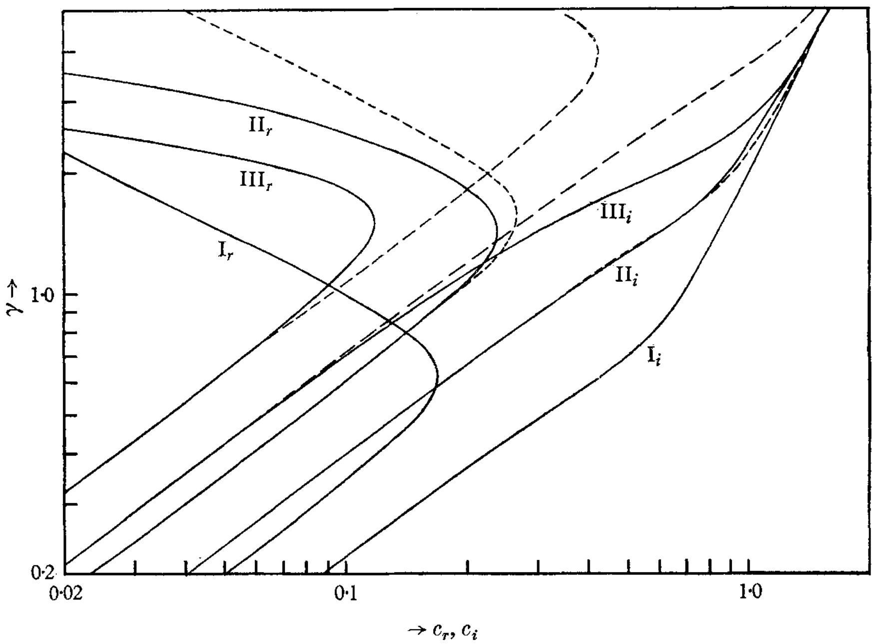


FIGURE 1. The curves describe the real and the imaginary parts $c_r, c_i$ of $c(\gamma)$ for the values $\eta = -0{\cdot}1$ (I), $\eta = -1$ (II), $\eta = -10$ (III). The dashed lines are obtained when the approximation $\lambda tg\frac{1}{2}\lambda \approx \frac{1}{2}\lambda^2$ is used in equation (4.8). In the case (I) the dashed lines coincide with the exact results.

The square root in (4.7) has to be chosen so that its real part is positive. Besides (4.7), solutions which are antisymmetric in $z$ are possible. We shall restrict our attention to the lowest symmetric mode because the growth rates for the higher modes are smaller. The second relation in (4.6),

$$c^2\left(\gamma^2 - \frac{4\lambda^2}{\gamma^2c^2}\right)^{\frac{1}{2}} + \frac{\gamma^2}{2} = 0, \tag{4.9}$$

together with (4.8) leads to two complex conjugate roots $c(\gamma)$ which represent the dispersion relation. The real and the imaginary parts are plotted in figure 1 for different values of $\eta$. For positive values of $\eta$ the real part of $c(\gamma)$ is negative. The results show that, for values of $\gamma$ somewhat larger than one, the influence of the top and bottom boundaries vanishes, and the solution becomes

$$c = \pm i(\tfrac{1}{2}\gamma)^{\frac{1}{2}}, \tag{4.10}$$

as in the case of a free shear flow profile (4.5) governed by the Rayleigh stability equation. In the case when either $\eta^2 \ll 1$ or $\gamma \ll \eta^{-\frac{1}{2}}$, the approximation

$$\lambda tg(\tfrac{1}{2}\lambda) \approx \tfrac{1}{2}\lambda^2 \tag{4.11}$$

holds. This approximation corresponds to the approximation used in the derivation of (3.3) since the perturbation approach (3.2) can be used not only when (3.1) holds, but also when the $\phi$-dependence of $\mathbf{q}_0$ is small compared with the $s$-dependence. The corresponding dispersion relation has also been plotted in figure 1, and the comparison with the exact results shows that the approximation (4.11) can be used to a larger extent than might be expected from its derivation. This is true in particular for the imaginary part of $c$, which suggests that (3.3) in the limit of vanishing $E$ provides a good approximation for the determination of the critical Rossby number even in the case when the condition (3.1) is not satisfied. Thus the mechanism of the constraint of the changing depth can be described generally as the stretching and compressing of vortex tubes.

## 5. Examples for the instability of circular shear flow

In order to obtain simple analytical solutions of (2.14) we assume that $f$ is piecewise constant or proportional to $s^{-2}$. We neglect the effects of side-walls and take $n_z^T = 1$. Cases with $n_z^T < 1$ can be derived from the case $n_z^T = 1$ by replacing $m$ with $m(n_z^T)^{-1}$ according to (2.14).

A profile which is representative for a shear layer is given by

$$f = \begin{cases} -1 \quad \text{for} \quad s \leqslant s_1, \\ \left(s_2^2 + s_1^2 - \dfrac{2s_1^2 s_2^2}{s^2}\right)(s_2^2 - s_1^2)^{-1} \quad \text{for} \quad s_1 < s < s_2, \\ 1 \quad \text{for} \quad s_2 \leqslant s. \end{cases} \tag{5.1}$$

Since $(3/s)f' + f''$ vanishes in all three regions, (2.15) can be solved easily:

$$\chi = \begin{cases} A_1 s^m \quad \text{for} \quad s \leqslant s_1, \\ A_2 s^m + B_2 s^{-m} \quad \text{for} \quad s_1 < s < s_2, \\ B_3 s^{-m} \quad \text{for} \quad s_2 \leqslant s. \end{cases} \tag{5.2}$$

The unknown coefficients in (5.2) are determined by requirements that the normal velocity or $\chi$, respectively, and the expression

$$(f-c)\frac{d}{ds}\chi - \chi\frac{d}{ds}f$$

obtained by integration of (2.14) are continuous at $s = s_1, s_2$. The four homogeneous equations for the four known coefficients are solvable when $c$ satisfies the characteristic equation

$$(mc-1)^2 = (m-1)^2 + \frac{4\gamma}{1-\gamma}\left(\frac{1-\gamma^m}{1-\gamma} - m\right), \tag{5.3}$$

where $\gamma$ is defined by $\gamma = (s_1/s_2)^2$. The case $m = 1$ is exceptional, since the right-hand side of (5.3) vanishes all for $\gamma$. Hence $c$ is real, and the profile (5.1) is always stable with respect to the disturbance with $m = 1$. For $m > 1$ the point of marginal stability is determined by the minimum of the right-hand side of (5.3) provided that this minimum is negative. In the limit when $\gamma$ approaches 1, the right-hand side has its minimum when

$$m \approx \frac{1\cdot594}{1-\gamma}[1-0\cdot842(1-\gamma)+\ldots]. \tag{5.4}$$

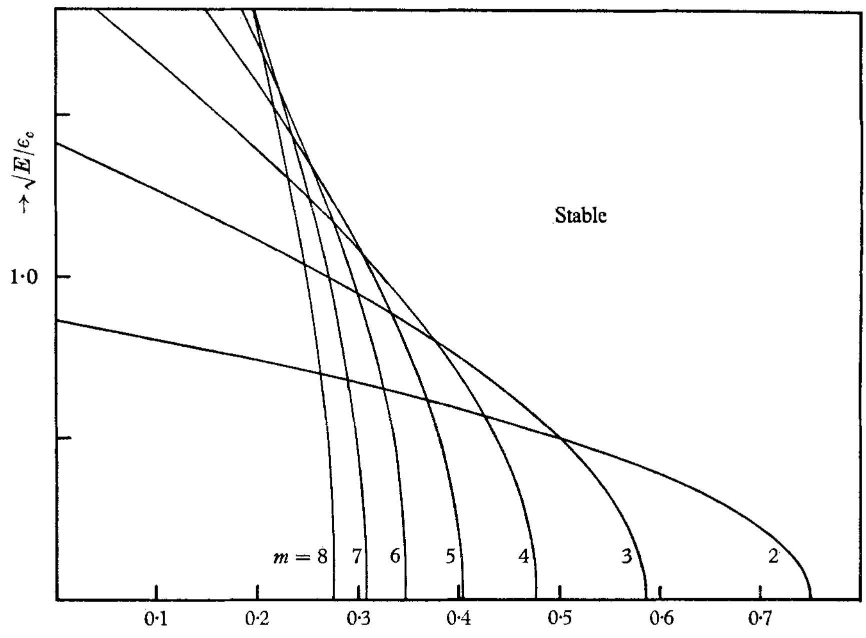


FIGURE 2. Stability diagram for the shear layer (5.1). The curves refer to instabilities with different azimuthal wave-number. The stationary flow corresponding to a certain $\gamma$ becomes unstable with respect to that instability which has the highest value of $\sqrt{E}/\epsilon_c$.

The corresponding value of the critical Rossby number is

$$\epsilon_c = \frac{2\sqrt{E}}{0\cdot805}(1-\gamma)[1+0\cdot186(1-\gamma)+\ldots]. \tag{5.5}$$

Relation (5.4) shows that the wavelength of the most unstable disturbance is of the same order as the width of the shear layer. For lower values of $m$ which are not covered by the asymptotic formula (5.5), the critical Rossby number is given in figure 2. No instability can occur for $\gamma < \frac{1}{4}$ corresponding to a ratio 2 of $s_2/s_1$. In the case of inclined boundaries, however, with $n_z^T < 1$, the mode with $m = 1$ may become unstable for values of $\gamma$ less than $\frac{1}{4}$.

Another case in which the stability problem can be solved easily is given by

$$f = \begin{cases} 0 \quad \text{for} \quad s \leqslant s_1, \\ \dfrac{s_1^2 s_2^2}{s_2^2 - s_1^2}\left(\dfrac{1}{s_1^2} - \dfrac{1}{s^2}\right) \quad \text{for} \quad s_1 < s < s_2, \\ \dfrac{s_2^2}{s^2} \quad \text{for} \quad s_2 \leqslant s. \end{cases} \tag{5.6}$$

The solution for $\chi$ in the form (5.2) leads to a characteristic equation for $c$

$$(mc-\tfrac{1}{2}m)^2 = \tfrac{1}{4}(m-2)^2+\frac{\gamma}{1-\gamma}\left(\frac{1-\gamma^{m-1}}{1-\gamma}-(m-1)\right). \tag{5.7}$$

The comparison with (5.3) shows that the right-hand side of (5.7) becomes identical with the right-hand side of (5.3) when multiplied by a factor 4 and when $m$ is replaced by $(m+1)$. Hence, the shear flow of the form (5.6) is stable with respect to disturbances with $m = 1, 2$, and figure 2 applies in this case with $\frac{1}{2}\epsilon_c$ and $(m+1)$ in place of $\epsilon_c$ and $m$. Again, in the case of inclined boundaries, modes with $m = 1, 2$ may become unstable when $n_z^T$ is sufficiently small.

## 6. Comparison with experimental observations

Critical Rossby numbers for the instability of a shear layer in a rotating system have been measured recently by Hide (1967). His experiment consists of a cylindrical container of height $L$ and radius $s_0$ filled with water and rotating about its axis with the angular velocity $\Omega$. A coaxial circular disk of radius $s_1$ which is part of the top plate of the cylinder is rotating at the rate $\Omega(1+4\epsilon)$. The experiment corresponds approximately to the case considered in Stewartson's (1957) paper which we have mentioned in the introduction. The experiment is not ideal for the purpose of testing the validity of theory described in § 2. The leading term of Stewartson's solution

$$f = \begin{cases} 1-\exp\{-(4/E)^{\frac{1}{4}}(s_0-s)\}+\ldots & \text{for} \quad s < s_0, \\ -1+\exp\{-(4/E)^{\frac{1}{4}}(s-s_0)\}+\ldots & \text{for} \quad s > s_0 \end{cases} \tag{6.1}$$

gives a thickness of the order $E^{\frac{1}{4}}$ for the shear layer. Hence the viscous dissipation in the interior becomes of the same order as the dissipative term induced by the Ekman boundary layer. Numerical calculations, however, by Esch (1957) for a profile similar to (6.1) indicate that the interior dissipation amounts only to a few percent of the boundary layer dissipation for the observed wave-numbers. Even the inviscid Rayleigh equation (2.18) does not allow a simple analytical solution for the profile (6.1) and for this reason we assume that the characteristic function $c(\alpha)$ of (2.18) can be represented by the characteristic functions of profiles similar to (6.1) as, for example,

$$f = \tanh\left\{\left(\frac{4}{E}\right)^{\frac{1}{4}}(s-s_0)\right\}, \quad \text{or} \tag{6.2}$$

$$f = \begin{cases} \dfrac{s-s_0}{|s-s_0|} & \text{for} \quad |s-s_0| \geqslant \left(\dfrac{E}{4}\right)^{\frac{1}{4}}\dfrac{1}{0\cdot 64}, \\ (s-s_0)\left(\dfrac{4}{E}\right)^{\frac{1}{4}}0\cdot 64 & \text{for} \quad |s-s_0| < \left(\dfrac{E}{4}\right)^{\frac{1}{4}}\dfrac{1}{0\cdot 64}. \end{cases} \tag{6.3}$$

The profile (6.3) has been normalized in such a way that the growth rates $\alpha c(\alpha)$ of (6.2) and (6.3) agree approximately (see Drazin & Howard 1962). The profile (6.3) is particularly suitable for our purpose since it corresponds to the limit $\gamma \to 1$ of the profile (5.1). This allows us to include the effects due to the circular geometry—which are important in Hide's experiment—by assuming that

$c(m)$ for the shear layer depends similarly on the thickness parameter $\gamma$ as $c(m)$ for the profile (5.1). Due to the assumptions involved we cannot expect more than qualitative agreement between theory and experimental observations. The observed instabilities appear as waves travelling along the shear layer. They are very similar to those shown in the paper by Hide & Titman (1967) and agree

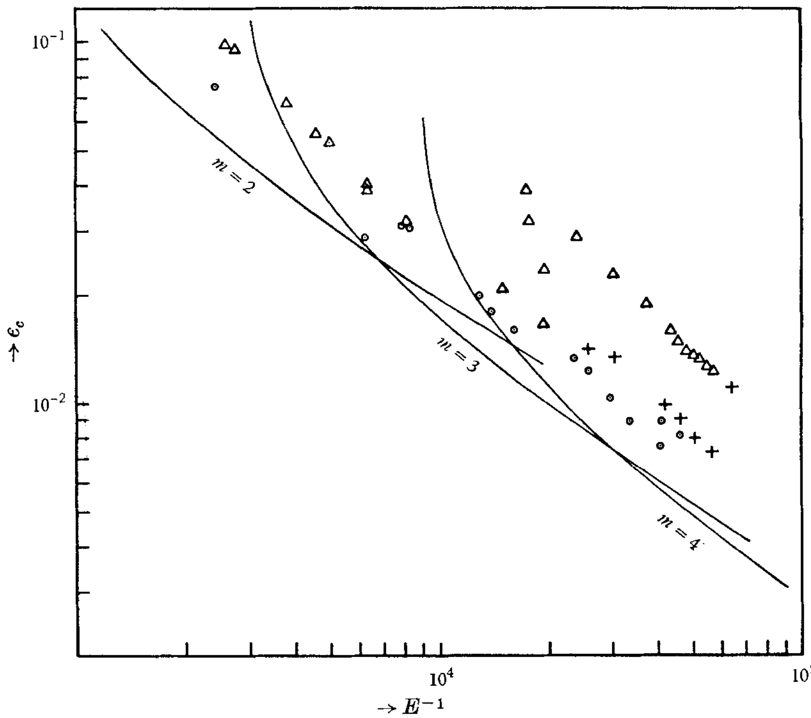


FIGURE 3. Experimental data by R. Hide in comparison with theoretical predictions. The observation of the axisymmetric regime is indicated by $\odot$; 3 waves ($m = 3$) by $\triangle$; 4 waves ($m = 4$) by $+$.

qualitatively with the theoretical description, at least in the case when the inner disk is rotating faster than the cylinder. Measured critical Rossby numbers and theoretical predictions are plotted in figure 3. The theoretical curves differ slightly from those corresponding to figure 2 because the constraint of the side-walls has been taken into account. As power law fitting the data, Hide gives

$$\epsilon_c = 32{\cdot}4\, E^{0{\cdot}77},$$

which is comparable with the asymptotic theoretical law derived from (5.5)

$$\epsilon_c = 15{\cdot}6 E^{\frac{3}{4}}.$$

The reasonable agreement suggests that a quantitative agreement might be expected in cases more suitable for the comparison between experiment and theory.

In many experimental situations, as for example in Hide's experiment, the amplitude of the Ekman boundary-layer flow is of the order of the Rossby number. It is known from the work of Faller (1963) and others that at critical Rossby numbers of the order $\sqrt{E}$ boundary-layer instabilities occur. Owing to there stricting influence of the rigid boundary, however, the Ekman–Reynolds number $\epsilon/\sqrt{E}$ is of the order 50 and larger in some cases. For this reason and because the scale of instabilities caused by the shear in the Ekman layer is very different from the scale characteristic for the interior, it seems justified to separate the two mechanisms of instability. Hence, we expect that the theory described in the previous sections is applicable in cases when the modification of the velocity field (1.1) in the Ekman layer is of the order of the Rossby number.

## 7. Conclusion

The stability theory for inviscid plane parallel shear flow is one of the most intensively studied fields in fluid dynamics. Owing to viscous dissipation and because of the fact that instabilities in the physical situation occur subcritically as disturbances of finite amplitude, there has been little comparison between theory and experiment. Both obstacles can be eliminated when the theory is applied to shear flow occurring in an appropriate rotating system of constant depth. The dissipative effects which occur mainly in the Ekman boundary layer can be taken into account without changing the form of the inviscid stability equation. That subcritical instabilities of finite amplitude do not occur in rotating systems of constant depth is suggested by the fact that the linear theory can predict adequately the critical Rossby number. A physical explanation for this fact is that three-dimensional motions usually associated with stabilities of finite amplitude are prohibited according to the Taylor–Proudman theorem.

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